(2) Test the Convergence of

(3) QN. → $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^3}$

Ans. → The given series is

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^3} = \frac{1}{2(\log 2)^3} + \frac{1}{3(\log 3)^3} + \frac{1}{4(\log 4)^3} + \cdots$$

$$+ \frac{1}{n(\log n)^3} + \cdots \text{ to } \infty.$$

Hence, Let $f(x) = \frac{1}{n(\log n)^3}$ for all $n > 1$

Case (i) when, $3 > 0$ Then $f(n)$ Continuously decreases as n increases.

Hence, by taking $a = 2$ in Cauchy's Condensation Test $\sum 2^n f(2^n)$

i.e. $\sum 2^n \frac{1}{2^n (\log 2^n)^3}$ Converges or diverges.

i.e. $\sum f(n)$ converges or diverges according as

$$\sum \frac{1}{n^3 (\log 2)^3} \text{ converges and diverges.}$$

But $\frac{1}{(\log 2)^3}$ is constant

Hence, $\sum f(n)$ converges or diverges according as $\sum \frac{1}{n^3}$ is converges or diverges.

$\because$ we know that,

$\sum \frac{1}{n^3}$ is converges when $p>1$

and diverges when $p \leq 1$

Hence, from Comparison Test

$$\sum \frac{1}{n(\log n)^3} \text{ diverges if } 3 \leq 0$$

Hence, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^3}$ is convergent when $3>1$

and divergent when $3 \leq 1$.

ⓐ Q.No $\rightarrow \sum_{n=2}^{\infty} \frac{n+2}{n^2-1} \cdot \frac{1}{(\log n)^{11/10}}$

Ans. $\rightarrow$ Let the general term of the given series be denoted by $f(n)$.

$$\therefore f(n) = \frac{n+2}{n^2-1} \cdot \frac{1}{(\log n)^{11/10}} = \boxed{\frac{n(1+\frac{2}{n})}{n^2(1-\frac{1}{n^2})} \cdot \frac{1}{(\log n)^{11/10}}}$$

Let us consider an Auxiliary series $\sum \phi(n)$, whose n^{th} term $\phi(n) = \frac{1}{n} \cdot \frac{1}{(\log n)^{11/10}}$

$$\text{Now, } = \frac{f(n)}{\phi(n)} = \frac{\cancel{\frac{1}{n} \cdot \frac{1}{(\log n)^{11/10}}}}{} = \frac{\frac{n+2}{n^2-1} \cdot \frac{1}{(\log n)^{11/10}}}{\frac{1}{n} \cdot \frac{1}{(\log n)^{11/10}}}$$

$$= \frac{n+2}{n^2-1} \cdot \frac{1}{\cancel{(\log n)^{11/10}}} \times \frac{n}{1} \cdot \frac{\cancel{(\log n)^{11/10}}}{1}$$

$$= \frac{n^2+2n}{n^2-1} = \frac{\cancel{n^2}\left(1+\frac{2}{n}\right)}{\cancel{n^2}\left(1-\frac{1}{n^2}\right)}$$

$$\lim_{n\to\infty} \frac{f(n)}{\phi(n)} = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)}{\left(1-\frac{1}{n^2}\right)} = \frac{\left(1+\frac{2}{\infty}\right)}{\left(1-\frac{1}{\infty^2}\right)} = \frac{1+0}{1-0} = \frac{1}{1}$$

$= 1$, which is finite and non zero

Hence, by Comparison test $\Sigma f(n)$ and $\Sigma \phi(n)$ will behave alike Clearly $\phi(n)$ is a positive decreasing function of n, because $n(\log n)^{11/10}$ increase with n.

Let us suppose that a +ve integer >1

$$\therefore a^n \phi(a)^n = \cancel{a^n} \cdot \frac{1}{\cancel{a^n}(\log a^n)^{11/10}}$$

$$= \frac{1}{(n \log a)^{11/10}}$$

$$= \frac{1}{n^{11/10}(\log a)^{11/10}} \Rightarrow \Sigma a^n \phi(a^n) = \frac{1}{(\log a)^{11/10}} \Sigma \frac{1}{n^{11/10}}$$

$\therefore \Sigma \frac{1}{n^{11/10}}$ is Convergent as $p = \frac{11}{10} > 1$, when Compared with $\Sigma \frac{1}{n^p}$

$\therefore \Sigma a^n \phi(a)^n$ is Convergent.

Hence, from Cauchy's Condensation Test the given series $\Sigma \phi(n)$ also Convergent

Thus from Comparison test $\Sigma f(n)$ is also Convergent.

Q. No. → Test the Convergence of $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$

Ans. →

$$\text{Here, } f(n) = \frac{1}{(\log n)^{\log n}}$$

Here, $f(n)$ is a positive decreasing function of n, because $(\log n)^{\log n}$ increase as n increases.

Let us suppose that a be any +ve integer > 1.

$$\therefore a^n f(a^n) = a^n \cdot \frac{1}{(\log a^n)^{\log a^n}}$$

$$= a^n \cdot \frac{1}{(n \log a)^{n \log a}}$$

$$\cancel{= a^n \cdot \frac{1}{n^n \log}}$$

$$= a^n \cdot \frac{1}{n^{n \log a} (\log a)^{n \log a}}$$

Let us take $a = 3$ and since $\log a$, that is, $\log 3 > 1$, we have,

$$\frac{1}{\log 3} < 1$$

and, So that,

$$\frac{1}{n^n \log 3} < \frac{1}{n^n} \text{ and } \frac{1}{(\log 3)^n \log 3} < 1.$$

$$\therefore a^n f(a^n) < \frac{3^n}{n^n} \leq \left(\frac{3}{4}\right)^n, \text{ where, } n \geq 4.$$

But $\Sigma \left(\frac{3}{4}\right)^n$ is Convergent, as it is in G.P. where c.r. is $\frac{3}{4}$.

~~Hence, $\Sigma a^n f(a^n)$ is Convergent~~

$\therefore$ From Cauchy's Condensation test $\Sigma f(n)$ ~~is Con~~ that is $\Sigma a^n f(a^n)$ is Convergent.

7. Q.No.→ Test for Convergence, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \log n}$

Ans.→ Let the general term of the given series be denoted by $f(n)$

Here, $f(n) = \frac{1}{\sqrt{n} \log n}$

Here $f(n)$ is +ve and decreasing, because denominator increases with n.

Let us suppose that a +ve integer >1

$$\therefore a^n f(a^n) = a^n \cdot \frac{1}{\sqrt{a^n} \log a^n} = \frac{\sqrt{a^n}}{n \log a}$$

Let the given term be denoted by u_n

$$u_n = \frac{\sqrt{a^n}}{n \log a}$$

Replacing n by $n+1$, we have

$$u_{n+1} = \frac{\sqrt{a^{n+1}}}{(n+1)\log a}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{\sqrt{a^n}\sqrt{a}}{(n+1)\log a} \times \frac{n\log a}{\sqrt{a^n}} = \frac{n}{n+1}\cdot\sqrt{a}$$

$$= \frac{n}{n(1+1/n)}\cdot\sqrt{a}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{1}{(1+1/n)}\sqrt{a} = \frac{1}{(1+1/\infty)}\sqrt{a} = \frac{1}{1+0}\cdot\sqrt{a} = \sqrt{a}$$

$\therefore$ From D' Alembert's ratio test, Σu_n that is, $\Sigma a^n f(a^n)$ is divergent, ~~when~~ and the given series is divergent, when $\sqrt{a} > 1$ as $a > 1$

Hence, from Cauchy's Condensation test, the given series is also divergent.